Assignment 5 for MATH4220

April 10, 2018

In Chapter 5, we will cover section 5.1, 5.2, 5.3, 5.4, 5.6. **No need to hand in.**
Exercise 5.1: 2, 3, 4, 5, 8, 9.
Exercise 5.2: 1(a), 1(b), 1(c), 1(d), 1(f), 2, 4, 5, 7, 9, 10.
Extra: Find out the solution for the following problem:

$$u_{tt} - 4u_{xx} = 0,\ u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin^2(\pi x), u_t(x,0) = x(1-x)$$

Exercise 5.3: 3, 5(a), 6, 8, 9, 10, 12, 13.
Exercise 5.4: 1, 2, 3, 4, 5, 6, 7, 8.
Exercise 5.6: 1, 2, 5, 8, 11.

Exercise 5.1

2. Let $\phi(x) \equiv x^2$ for $0 \le x \le 1 = l$.
   (a) Calculate its Fourier sine series.
   (b) Calculate its Fourier cosine series.

4. Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

5. Given the Fourier sine series of $\phi(x) \equiv x$ on $(0, l)$. Assume that the series can be integrated term by term, a fact that will be shown later.
   (a) Find the Fourier cosine series of the function $x^2/2$. Find the constant of integration that will be the first term in the cosine series.
   (b) Then by setting $x = 0$ in your result, find the *sum* of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

8. A rod has length $l = 1$ and constant $k = 1$. Its temperature satisfies the heat equation. Its left end is held at temperature 0, its right end at temperature 1. Initially (at $t = 0$) the temperature is given by

$$\phi(x) = \begin{cases} \frac{5x}{2} & 0 < x < \frac{2}{3} \\ 3 - 2x & \frac{2}{3} < x < 1. \end{cases}$$

   Find the solution, including the coefficients. (*Hint:* First find the equilibrium solution $U(x)$, and then solve the heat equation with initial condition $u(x,0) = \phi(x) - U(x)$.)

9. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$, with the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and the initial conditions $u(x,0) = 0,\ u_t(x,0) = \cos^2 x$. (*Hint:* See (4.2.7).)

Exercise 5.2

2. Show that $\cos x + \cos \alpha x$ is periodic if α is a rational number. What is its period?

5. Show that the Fourier sine series on $(0, l)$ can be derived from the full Fourier series on $(-l, l)$ as follows. Let $\phi(x)$ be any (continuous) function on $(0, l)$. Let $\tilde{\phi}(x)$ be its odd extension. Write the full series for $\tilde{\phi}(x)$ on $(-l, l)$. [Assume that its sum is $\tilde{\phi}(x)$.] By Exercise 4, this series has only sine terms. Simply restrict your attention to $0 < x < l$ to get the sine series for $\phi(x)$.

8. (a) Prove that differentiation switches even functions to odd ones, and odd functions to even ones.
   (b) Prove the same for integration provided that we ignore the constant of integration.

10. (a) Let $\phi(x)$ be a continuous function on $(0, l)$. Under what conditions is its *odd* extension also a continuous function?
    (b) Let $\phi(x)$ be a differentiable function on $(0, l)$. Under what conditions is its *odd* extension also a differentiable function?
    (c) Same as part (a) for the *even* extension.
    (d) Same as part (b) for the *even* extension.

Exercise 5.3

3. Consider $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0, t) = 0$, $u_x(l, t) = 0$ and the initial conditions $u(x, 0) = x$, $u_t(x, 0) = 0$. Find the solution explicitly in series form.

5(a). Show that the boundary conditions $u(0, t) = 0, u_x(l, t) = 0$ lead to the eigenfunctions $(\sin(\pi x/2l)$, $\sin(3\pi x/2l)$, $\sin(5\pi x/2l)$, $\cdots)$.

6. Find the complex eigenvalues of the first-derivative operator d/dx subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?

8. Show directly that $(-X_1' X_2 + X_1 X_2')|_a^b = 0$ if both X_1 and X_2 satisfy the same Robin boundary condition at $x = a$ and the same Robin boundary condition at $x = b$.

9. Show that the boundary conditions

$$X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)$$

   on an interval $a \le x \le b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.

12. Prove *Green's first identity*: For every pair of functions $f(x)$, $g(x)$ on (a, b),

$$\int_a^b f''(x) g(x) dx = -\int_a^b f'(x) g'(x) dx + f' g \Big|_a^b.$$

13. Use Greens first identity to prove Theorem 3. (*Hint:* Substitute $f(x) = X(x) = g(x)$, a real eigenfunction.)

Exercise 5.4

1. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ is a geometric series.
   (a) Does it converge pointwise in the interval $1 < x < 1$?
   (b) Does it converge uniformly in the interval $1 < x < 1$?
   (c) Does it converge in the L^2 sense in the interval $1 < x < 1$? (*Hint:* You can compute its partial sums explicitly.)

2. Consider any series of functions on any finite interval. Show that if it converges uniformly, then it also converges in the L^2 sense and in the pointwise sense.

3. Let γ_n be a sequence of constants tending to ∞. Let $f_n(x)$ be the sequence of functions defined as follows: $f_n(\frac{1}{2}) = 0, f_n(x) = \gamma_n$ in the interval $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2})$, let $f_n(x) = -\gamma_n$ in the interval $(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]$ and let $f_n(x) = 0$ elsewhere. Show that:

   (a) $f_n(x) \to 0$ pointwise.
   (b) The convergence is not uniform.
   (c) $f_n(x) \to 0$ in the L^2 sense if $\gamma_n = n^{1/3}$.
   (d) $f_n(x)$ does not converge in the L^2 sence if $\gamma_n = n$.

4. Let

$$g_n(x) = \begin{cases} 1 \text{ in the interval } \left[\frac{1}{4} - \frac{1}{n^2}, \frac{1}{4} + \frac{1}{n^2}\right) & \text{for odd } n \\ 1 \text{ in the interval } \left[\frac{3}{4} - \frac{1}{n^2}, \frac{3}{4} + \frac{1}{n^2}\right) & \text{for even } n \\ 0 & \text{for all other } x. \end{cases}$$

   Show that $g_n(x) \to 0$ in the L^2 sense but that $g_n(x)$ does not tend to zero in the pointwise sense.

5. Let $\phi(x) = 0$ for $0 < x < 1$ and $\phi(x) = 1$ for $1 < x < 3$.

   (a) Find the first four nonzero terms of its Fourier cosine series explicitly.
   (b) For each $x (0 \leq x \leq 3)$, what is the sum of this series?
   (c) Does it converge to $\phi(x)$ in the L^2 sense? Why?
   (d) Put $x = 0$ to find the sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots.$$

6. Find the sine series of the function cos x on the interval $(0, \pi)$. For each x satisfying $-\pi \leq x \leq \pi$, what is the sum of the series?

7. Let

$$\phi(x) = \begin{cases} -1 - x & -1 < x < 0 \\ 1 - x & 0 < x < 1. \end{cases}$$

   (a) Find the full Fourier series of $\phi(x)$ in the interval $(-1, 1)$.
   (b) Find the first three nonzero terms explicitly.
   (c) Does it converge in the mean square sense?
   (d) Does it converge pointwise?
   (e) Does it converge uniformly to $\phi(x)$ in the interval $(-1, 1)$?

Exercise 5.6

1. (a) Solve as a series the equation $u_t = u_{xx}$ in $(0, 1)$ with $u_x(0, t) = 0$, $u(1, t) = 1$, and $u(x, 0) = x^2$. Compute the first two coefficients explicitly.
   (b) What is the equilibrium state (the term that does not tend to zero)?

2. For problem (1), complete the calculation of the series in case $j(t) = 0$ and $h(t) = e^t$.

5. Solve $u_{tt} = c^2 u_{xx} + e^t \sin 5x$ for $0 < x < \pi$, with $u(0, t) = u(\pi, t) = 0$ and the initial conditions $u(x, 0) = 0$, $u_t(x, 0) = \sin 3x$.

8. Solve $u_t = ku_{xx}$ in $(0, l)$, with $u(0, t) = 0$, $u(l, t) = At$, $u(x, 0) = 0$, where A is a constant.